CLASSIFICATION OF $\mathbb{C}P^2$-MULTIPLICATIVE HIRZEBRUCH GENERA

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Abstract. The short article [1] states results on $\mathbb{C}P(2)$-multiplicative Hirzebruch genera. The aim of the following text is to give a proof of Theorem 3 from [1]. This proof uses only the technique of functional differential equations.

1. Preliminaries

Let R be a commutative torsion-free ring with unity and no zero divisors, and let $L_f : \Omega_U \to R$ be the Hirzebruch genus determined by the series $f(x) = x + \sum_{k=1}^{\infty} f_k \frac{x^{k+1}}{(k+1)!}$, where $f_k \in R$.

A Hirzebruch genus $L_f : \Omega_U \to R$ is called $\mathbb{C}P(2)$-*multiplicative*, if we have $L_f[M] = L_f[\mathbb{C}P(2)]L_f[B]$ for any bundle of stably complex manifolds $M \to B$ with fiber $\mathbb{C}P(2)$ and structure group G such that $U^*(BG)$ is torsion-free. From the localization theorem for the universal toric genus (see [2]) for the standard action of the torus T^3 on the complex projective plane $\mathbb{C}P(2)$, theorem holds:

Theorem 1. *A genus L_f is $\mathbb{C}P(2)$-multiplicative if and only if $f(x)$ solves the functional equation*

$$\frac{1}{f(t_1-t_2)f(t_1-t_3)} + \frac{1}{f(t_2-t_1)f(t_2-t_3)} + \frac{1}{f(t_3-t_1)f(t_3-t_2)} = C, \quad C \in R. \tag{1}$$

In [3] it was shown with the help of equation (1) that for bundles of *oriented* manifolds the universal $\mathbb{C}P(2)$–multiplicative genus is determined by the signature of the manifold. We have $C = L_f[\mathbb{C}P(2)] = \frac{3f_1^2 - f_2}{2}$.

2. Theorem

In [1] the following theorem is proposed. Its proof is given in the next section.

Theorem 2. *Let L_f be a $\mathbb{C}P(2)$-multiplicative genus.*

If $L_f[\mathbb{C}P(2)] \neq 0$, then L_f is the two-parametric Todd genus, and

$$f(x) = \frac{e^{\alpha x} - e^{\beta x}}{\alpha e^{\alpha x} - \beta e^{\beta x}}, \quad f_1 = -(\alpha+\beta), \quad f_2 = 2\alpha\beta + f_1^2, \quad f_3 = 4f_1f_2 - 3f_1^3. \tag{2}$$

If $L_f[\mathbb{C}P(2)] = 0$, then it is a two-parametric case of general elliptic genus in the terminology of [4], *and*

$$f(x) = -\frac{2\wp(x) + \frac{a^2}{2}}{\wp'(x) - a\wp(x) + b - \frac{a^3}{4}}. \tag{3}$$

Here $\wp$ and $\wp'$ are Weierstrass functions of the elliptic curve with parameters $g_2 = -\frac{1}{4}(8b - 3a^3)a$, $g_3 = \frac{1}{24}(8b^2 - 12a^3b + 3a^6)$, and discriminant $\Delta = -b^3(3b - a^3)$. The parameters a and b are related to the coefficients of the series $f(x)$ by $f_1 = -a$, $f_2 = 3a^2$, $f_3 = 12b - 9a^3$.

The genus determined by $f(x)$ as in (3) was first introduced in [5].

3. Proof

The proof of the theorem follows as a compilation of theorem 1 with the following three lemmas, each given with its own proof.

For convenience set $q(x)=\frac{1}{f(x)}$. Denote $x=t_1-t_2$, $y=t_2-t_3$. Equation (1) takes the form

$$q(x)q(x+y)+q(-x)q(y)+q(-x-y)q(-y)=C. \tag{4}$$

Lemma 3. *The function $q(x)=\frac{1}{f(x)}$, where*

$$f(x)=\frac{e^{\alpha x}-e^{\beta x}}{\alpha e^{\alpha x}-\beta e^{\beta x}}$$

satisfies the functional equation (4) for $C=\alpha^2+\alpha\beta+\beta^2$.

Proof. The proof is a straightforward substitution, namely, equation (4) takes the form

$$\frac{(\alpha e^{\alpha x}-\beta e^{\beta x})(\alpha e^{\alpha(x+y)}-\beta e^{\beta(x+y)})}{(e^{\alpha x}-e^{\beta x})(e^{\alpha(x+y)}-e^{\beta(x+y)})}+\frac{(\alpha e^{-\alpha x}-\beta e^{-\beta x})(\alpha e^{\alpha y}-\beta e^{\beta y})}{(e^{-\alpha x}-e^{-\beta x})(e^{\alpha y}-e^{\beta y})}+\frac{(\alpha e^{-\alpha(x+y)}-\beta e^{-\beta(x+y)})(\alpha e^{-\alpha y}-\beta e^{-\beta y})}{(e^{-\alpha(x+y)}-e^{-\beta(x+y)})(e^{-\alpha y}-e^{-\beta y})}=C,$$

which after multiplication of the nominator and denominator by a relevant factor becomes

$$\frac{(\alpha e^{\alpha x}-\beta e^{\beta x})(\alpha e^{\alpha(x+y)}-\beta e^{\beta(x+y)})}{(e^{\alpha x}-e^{\beta x})(e^{\alpha(x+y)}-e^{\beta(x+y)})}+\frac{(\beta e^{\alpha x}-\alpha e^{\beta x})(\alpha e^{\alpha y}-\beta e^{\beta y})}{(e^{\alpha x}-e^{\beta x})(e^{\alpha y}-e^{\beta y})}+\frac{(\beta e^{\alpha(x+y)}-\alpha e^{\beta(x+y)})(\beta e^{\alpha y}-\alpha e^{\beta y})}{(e^{\alpha(x+y)}-e^{\beta(x+y)})(e^{\alpha y}-e^{\beta y})}=C.$$

Bringing to a common factor one gets

$$\begin{aligned}&(\alpha e^{\alpha x}-\beta e^{\beta x})(e^{\alpha y}-e^{\beta y})(\alpha e^{\alpha(x+y)}-\beta e^{\beta(x+y)})+(\beta e^{\alpha x}-\alpha e^{\beta x})(\alpha e^{\alpha y}-\beta e^{\beta y})(e^{\alpha(x+y)}-e^{\beta(x+y)})+\\&\quad+(e^{\alpha x}-e^{\beta x})(\beta e^{\alpha y}-\alpha e^{\beta y})(\beta e^{\alpha(x+y)}-\alpha e^{\beta(x+y)})=C(e^{\alpha x}-e^{\beta x})(e^{\alpha y}-e^{\beta y})(e^{\alpha(x+y)}-e^{\beta(x+y)}).\end{aligned}$$

Now this expression is available for term-by-term check, like at $e^{2\alpha(x+y)}$ we have

$$\alpha^2+\alpha\beta+\beta^2=C$$

and the same for all other coefficients. $\square$

Lemma 4. *The function $q(x)=\frac{1}{f(x)}$, where*

$$f(x)=-\frac{2\wp(x)+\frac{a^2}{2}}{\wp'(x)-a\wp(x)+b-\frac{a^3}{4}}$$

with parameters $g_2=-\frac{1}{4}(8b-3a^3)a$ and $g_3=\frac{1}{24}(8b^2-12a^3b+3a^6)$ of the Weierstrass $\wp$-function satisfies the functional equation (4) for $C=0$.

Proof. We have

$$q(x)=\frac{a}{2}-\frac{b}{2\wp(x)+\frac{a^2}{2}}-\frac{\wp'(x)}{2\wp(x)+\frac{a^2}{2}}.$$

For $C=0$ equation (4) after the substitution of $q(x)$ takes the form (here we take into account that $\wp$ is an even function and $\wp'$ is odd)

$$\begin{aligned}&\left(\frac{a}{2}-\frac{b}{2\wp(x)+\frac{a^2}{2}}-\frac{\wp'(x)}{2\wp(x)+\frac{a^2}{2}}\right)\left(\frac{a}{2}-\frac{b}{2\wp(x+y)+\frac{a^2}{2}}-\frac{\wp'(x+y)}{2\wp(x+y)+\frac{a^2}{2}}\right)+\\&\quad+\left(\frac{a}{2}-\frac{b}{2\wp(x)+\frac{a^2}{2}}+\frac{\wp'(x)}{2\wp(x)+\frac{a^2}{2}}\right)\left(\frac{a}{2}-\frac{b}{2\wp(y)+\frac{a^2}{2}}-\frac{\wp'(y)}{2\wp(y)+\frac{a^2}{2}}\right)+\\&\quad+\left(\frac{a}{2}-\frac{b}{2\wp(x+y)+\frac{a^2}{2}}+\frac{\wp'(x+y)}{2\wp(x+y)+\frac{a^2}{2}}\right)\left(\frac{a}{2}-\frac{b}{2\wp(y)+\frac{a^2}{2}}+\frac{\wp'(y)}{2\wp(y)+\frac{a^2}{2}}\right)=0.\end{aligned}$$

After bringing this expression to a common denominator we obtain that it is required to prove the relation

$$\left(\wp(y)+\frac{a^2}{4}\right)\left(\wp'(x)+b-a\left(\wp(x)+\frac{a^2}{4}\right)\right)\left(\wp'(x+y)+b-a\left(\wp(x+y)+\frac{a^2}{4})\right)\right)-$$
$$-\left(\wp(x+y)+\frac{a^2}{4}\right)\left(\wp'(x)-b+a\left(\wp(x)+\frac{a^2}{4}\right)\right)\left(\wp'(y)+b-a\left(\wp(y)+\frac{a^2}{4}\right)\right)+$$
$$+\left(\wp(x)+\frac{a^2}{4}\right)\left(\wp'(x+y)-b+a\left(\wp(x+y)+\frac{a^2}{4}\right)\right)\left(\wp'(y)-b+a\left(\wp(y)+\frac{a^2}{4}\right)\right)=0.$$

Consider the left part as a function of x where y is a parameter. It is a two-periodic function, it might have poles only in points comparable to $x=0$ and $x=-y$. Consider this function for $x=0$. We obtain 0 at $\frac{1}{x^3}$, while at $\frac{1}{x^2}$ we obtain

$$32(3a^4-8ab-4g_2)\wp(y)+12a^6-64a^3b+16a^2g_2-192g_3+64b^2,$$

which gives 0 after substituting g_2 and g_3. At $\frac{1}{x}$ we get

$$16(3a^4-8ab-4g_2)\wp'(y)$$

which gives 0 again. Therefore the left part has no poles in points comparable to $x=0$. As the equation is invariant under substitutions $(x\to y, y\to x, a\to -a, b\to -b)$ and $x\to y, y\to -x-y, a\to a, b\to b$, thus it has no poles comparable to $x=-y$. Therefore the left part of the expression, being a meromorphic function without poles, must be constant. Calculation of the free term at $x=0$ shows that this expression is equal to zero. □

Lemma 5. *The functional equation* (4) *does not have solutions other then stated in Lemmas* 3 *and* 4.

Proof. The series decomposition of (4) in y taking into account the initial conditions gives at y^k for $k=0$ the equation

$$q(x)^2-f_1q(-x)+q'(-x)=C, \tag{5}$$

for $k=1$ the derivative of (5) and for $k=2$ the equation

$$6q(x)q''(x)-Kq(-x)+(3f_1^2-2f_2)q'(-x)-3f_1q''(-x)+2q'''(-x)=0, \qquad K=3f_1^3-4f_1f_2+f_3. \tag{6}$$

Decomposing the equations (5) and (6) and taking into account initial conditions, we obtain from (5) at x^0 the relation $2C=3f_1^2-f_2$. Further from x^k in (5) and x^{k-2} in (6) for $k=2,3,4,5$ we get coinciding relations

$$\begin{aligned}
f_4&=15f_1^4-25f_1^2f_2+7f_1f_3+4f_2^2,\\
f_5&=15f_1^3f_2-15f_1^2f_3-10f_1f_2^2+6f_1f_4+5f_2f_3,\\
2f_6&=315f_1^6-945f_1^4f_2+345f_1^3f_3+660f_1^2f_2^2-93f_1^2f_4-290f_1f_2f_3-60f_2^3+18f_1f_5+32f_2f_4+20f_3^2,\\
f_7&=210f_1^5f_2-210f_1^4f_3-420f_1^3f_2^2+105f_1^3f_4+420f_1^2f_2f_3+140f_1f_2^3-35f_1^2f_5-112f_1f_2f_4-70f_1f_3^2-\\
&\quad-70f_2^2f_3+8f_1f_6+14f_2f_5+21f_3f_4.
\end{aligned}$$

At x^6 in (5) and at x^4 in (6) we get different relations:

$$\begin{aligned}
3f_8&=8505f_1^8-36855f_1^6f_2+14805f_1^5f_3+48300f_1^4f_2^2-4599f_1^4f_4-29820f_1^3f_2f_3-19320f_1^2f_2^3+1134f_1^3f_5+\\
&\quad+6552f_1^2f_2f_4+4095f_1^2f_3^2+10500f_1f_2^2f_3+1120f_2^4-222f_1^2f_6-980f_1f_2f_5-1470f_1f_3f_4-924f_2^2f_4-\\
&\quad-1155f_2f_3^2+33f_1f_7+80f_2f_6+140f_3f_5+84f_4^2,\\
19f_8&=53865f_1^8-233415f_1^6f_2+94500f_1^5f_3+304920f_1^4f_2^2-29862f_1^4f_4-188370f_1^3f_2f_3-121380f_1^2f_2^3+\\
&\quad+7497f_1^3f_5+41706f_1^2f_2f_4+25515f_1^2f_3^2+65730f_1f_2^2f_3+7000f_2^4-1476f_1^2f_6-6300f_1f_2f_5-\\
&\quad-9072f_1f_3f_4-5796f_2^2f_4-7140f_2f_3^2+216f_1f_7+516f_2f_6+861f_3f_5+504f_4^2.
\end{aligned}$$

Comparing this expressions for f_8 taking into account the expressions for f_4, f_5, f_6, f_7 and C we obtain the relation

$$CK^2=0.$$

Therefore we obtain either $C = 0$, which gives solution (3), or $K = 0$, which gives solution (2).

From (5) and initial conditions it follows that for given f_1, f_2, f_3 all f_k for $k \geqslant 4$ are uniquely defined, thus there are no other solutions. $\square$

References

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